Linear Algebra for Theoretical Neuroscience (Part 0)
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Current versions of all parts of this work can be found at http://www.neurotheory.columbia.edu/∼ken/math-notes. Please feel free to link to this site.

0 Reminders: Basics of Single-Variable Analysis

These linear algebra notes will focus on working in multi-dimensional spaces, *i.e.* equations in multiple variables. As background, we here review the basics of single-variable analysis. In practice, on a computer, all variables are discrete, so it will be helpful to understand the correspondence between continuous functions and discrete variables, *e.g.* integrals can be thought of as a limit of a discrete sum, and derivatives as the limit of a discrete difference.

Recall that the sum symbol $\sum_{i=0}^{N} x_i$ means to sum the values of x_i from $i = 0$ to $i = N$, *e.g.* $\sum_{i=0}^{2} x_i = x_0 + x_1 + x_2$.

0.1 Derivatives are Differences, Integrals are Sums, and the Fundamental Theorem of Calculus

Derivatives are differences: The derivative of a function f at a point x is the slope of the function at the point. It is defined as a limit of a difference between the values of $f(x)$ at two points separated by Δx, divided by Δx:

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{0.1}$$

The second derivative is the derivative of the first derivative, $\frac{d^2 f(x)}{dx^2} = \frac{d}{dx}\frac{df(x)}{dx}$. Similarly the third derivative is the derivative of the 2nd derivative, etc; the n^{th} derivative is written $\frac{d^n f(x)}{dx^n}$ or as $f^{'\cdots'}$ where there are n primes for the n^{th} derivative, *e.g.* the first, second, and third derivatives of $f(x)$ can be written $f'(x)$, $f''(x)$, $f'''(x)$ respectively. The n^{th} derivative can also be written $f^{(n)}(x)$.

Some commonly encountered derivatives include (below, $\log(x)$ means the natural log (log in base e), sometimes written $\ln(x)$):

$$\frac{dx^n}{dx} = nx^{n-1} \tag{0.2}$$

$$\frac{de^x}{dx} = e^x \tag{0.3}$$

$$\frac{d\log(x)}{dx} = \frac{1}{x} \tag{0.4}$$

$$\frac{d\cos(x)}{dx} = -\sin(x) \tag{0.5}$$

$$\frac{d\sin(x)}{dx} = \cos(x) \tag{0.6}$$

These example derivatives are derived in Section 0.4 below.

Integrals are sums: Recall that, for a function $f(x)$ of a variable x, the definite integral from A to B is the area under the curve defined by $f(x)$ from A to B. This is defined as a limit of a discrete sum:

$$\int_A^B dx\, f(x) \equiv \lim_{\Delta x \to 0} \sum_{i=0}^{\mathrm{Round}((B-A)/\Delta x)} f(A + i\Delta x)\Delta x \tag{0.7}$$

Note that there are $(B-A)/\Delta x$ values in the sum, which we round to the nearest integer, and they are weighted by Δx; thus the weighting times the number of values in the sum remains constant, allowing a limit to exist.

If $\int_A^B dx\, f(x) = F(B) - F(A)$ for some function $F(x)$, then the indefinite integral of $f(x)$ is defined as $\int dx\, f(x) = F(x) + C$, where C is an arbitrary constant. (Note that if $F(x)$ satisfies the definition of the indefinite integral, so too does $F(x) + C$ for arbitrary C.)

Some commonly encountered integrals (expressed here as indefinite integrals, but omitting the constant C) include:

$$\int dx\, x^n = \frac{x^{n+1}}{n+1} \qquad (n \neq -1) \tag{0.8}$$

$$\int dx\, \frac{1}{x} = \log(x) \tag{0.9}$$

$$\int dx\, e^x = e^x \tag{0.10}$$

$$\int dx\, \cos(x) = \sin(x) \tag{0.11}$$

$$\int dx\, \sin(x) = -\cos(x) \tag{0.12}$$

$$\tag{0.13}$$

Note that in each case the derivative of the outcome on the right gives the integrand, which follows from the fundamental theorem of calculus, to which we now turn.

The fundamental theorem of calculus: This states that the derivative and integral are inverses of each other in the following sense: if $F(x)$ is the indefinite integral of $f(x)$, then $f(x) = \frac{dF(x)}{dx}$. This can alternately be stated that the integral of the derivative of a function F is that function: $\int_A^B dx\, \frac{dF(x)}{dx} = F(B) - F(A)$, or $\int dx\, \frac{dF(x)}{dx} = F(x) + C$, where C is an arbitrary constant.

This can be seen from the definitions of integrals and derivatives:

$$\int_A^B dx\, \frac{dF(x)}{dx} = \lim_{\Delta x \to 0} \sum_{i=0}^{\mathrm{Round}((B-A)/\Delta x)} \frac{dF(A + i\Delta x)}{dx}\Delta x \tag{0.14}$$

$$= \lim_{\Delta x \to 0} \lim_{\Delta x' \to 0} \sum_{i=0}^{\mathrm{Round}((B-A)/\Delta x)} \frac{F(A + i\Delta x + \Delta x') - F(A + i\Delta x)}{\Delta x'}\Delta x \tag{0.15}$$

We can take the limit keeping $\Delta x = \Delta x'$, which then gives

$$= \lim_{\Delta x \to 0} \sum_{i=0}^{\mathrm{Round}((B-A)/\Delta x)} F(A + (i+1)\Delta x) - F(A + i\Delta x) \tag{0.16}$$

$$= \lim_{\Delta x \to 0} F\left(A + \left(\mathrm{Round}\left(\frac{B-A}{\Delta x}\right) + 1\right)\Delta x\right) - F(A + 0 * \Delta x) \tag{0.17}$$

$$= F(B) - F(A) \tag{0.18}$$

0.2 Taylor Series

Functions that are sufficiently well behaved can have their value near some value of x approximated by a Taylor series, which is a series in terms of the function's derivative:[1]

$$f(x+\epsilon) = f(x) + f'(x)\epsilon + \frac{1}{2!}f''(x)\epsilon^2 + \frac{1}{3!}f'''(x)\epsilon^3 + \ldots = \sum_0^\infty \frac{f^{(n)}(x)}{n!}\epsilon^n \tag{0.21}$$

Some specific examples of expansions about $x = 0$ that are frequently used are:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \qquad \text{(convergent for } |x| < 1) \tag{0.22}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots \qquad \text{(convergent for } |x| < 1) \tag{0.23}$$

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots \qquad \text{(convergent for } |x| < 1) \tag{0.24}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \qquad \text{(convergent for all } x) \tag{0.25}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \qquad \text{(convergent for all } x) \tag{0.26}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \qquad \text{(convergent for all } x) \tag{0.27}$$

0.3 The Chain Rule

When a function $f(y)$ depends on y, and y in turn depends on a second variable x, f can be differentiated with respect to x using the chain rule:[2]

$$\frac{df}{dx} = \frac{df}{dy}\frac{dy}{dx} \tag{0.28}$$

If f depends on multiple variables $y_i(x)$, $i = 1, \ldots, n$, then one can sum over the dependence on each variable:

$$\frac{df}{dx} = \frac{df}{dy_1}\frac{dy_1}{dx} + \frac{df}{dy_2}\frac{dy_2}{dx} + \ldots = \sum_{i=1}^N \frac{df}{dy_i}\frac{dy_i}{dx} \tag{0.29}$$

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[1] The form of the Taylor series can be derived as follows. Suppose there is an expansion

$$f(x+\epsilon) = a_0 + a_1\epsilon + a_2\epsilon^2 + \ldots = \sum_{i=0}^\infty a_i\epsilon^i \tag{0.19}$$

Then by setting ϵ to zero, one finds $a_0 = f(x)$. Taking one derivative with respect to ϵ gives

$$f'(x+\epsilon) = a_1 + 2a_2\epsilon + 3a_3\epsilon^2 + \ldots = \sum_{i=1}^\infty i a_i \epsilon^{i-1} \tag{0.20}$$

Setting ϵ to zero shows that $a_1 = f'(x)$. Repeating this process gives $f''(x) = 2a_2$, $f'''(x) = (3{\cdot}2)a_3$, $f^4(x) = (4{\cdot}3{\cdot}2)a_4$, etc., *i.e.* $f^{(n)}(x) = (n!)a_n$ or $a_n = f^{(n)}(x)/n!$. This does not tell when a Taylor series will converge, but tells the form it must take when it is convergent.

[2] The chain rule can be derived as follows. Given a function f(y(x)), then $\frac{df}{dx} = \lim_{\epsilon\to 0}\frac{f(y(x+\epsilon))-f(y(x))}{\epsilon}$. Now expanding $y(x+\epsilon)$ in a Taylor series gives $\frac{df}{dx} = \lim_{\epsilon\to 0}\frac{f(y(x)+y'(x)\epsilon+O(\epsilon^2))-f(y(x))}{\epsilon}$. (Here, $O(\epsilon^2)$ means terms with ϵ raised to the power 2 or higher; for a more rigorous definition see `https://en.wikipedia.org/wiki/Big_O_notation`). Expanding f in a Taylor series then gives $\frac{df}{dx} = \lim_{\epsilon\to 0}\frac{f(y(x))+f'(y(x))y'(x)\epsilon+O(\epsilon^2)-f(y(x))}{\epsilon} = f'(y(x))y'(x)$.

The chain rule allows us to generalize the example derivatives given above. For example,

$$\frac{de^{kx}}{dx} = \frac{de^{kx}}{d(kx)}\frac{d(kx)}{dx} = ke^{kx} \tag{0.30}$$

$$\frac{d\log(kx)}{dx} = \frac{d\log(kx)}{d(kx)}\frac{d(kx)}{dx} = k\frac{1}{kx} = \frac{1}{x} \tag{0.31}$$

(which can also be seen from $\log(kx) = \log(k) + \log(x)$)

$$\frac{d\sqrt{1+kx}}{dx} = \frac{d(1+kx)^{0.5}}{dx} = \frac{d(1+kx)^{0.5}}{d(1+kx)}\frac{d(1+kx)}{dx} = 0.5(1+kx)^{-0.5}k = \frac{k}{2\sqrt{1+kx}} \tag{0.32}$$

etc.

The chain rule also gives us the product rule:

$$\frac{d(f(x)g(x))}{dx} = \frac{d(f(x)g(x))}{df(x)}\frac{df(x)}{dx} + \frac{d(f(x)g(x))}{dg(x)}\frac{dg(x)}{dx} = f'(x)g(x) + f(x)g'(x) \tag{0.33}$$

Letting $g(x) = \frac{1}{h(x)}$, and noting $\frac{d(1/h(x))}{dx} = \frac{d(1/h(x))}{d(h(x))}\frac{dh(x)}{dx} = -\frac{h'(x)}{h(x)^2}$, we obtain the rule for differentiating the division of two functions

$$\frac{d\left(\frac{f(x)}{h(x)}\right)}{dx} = \frac{f'(x)}{h(x)} - f(x)\frac{h'(x)}{h(x)^2} = \frac{f'(x)h(x) - f(x)h'(x)}{h(x)^2} \tag{0.34}$$

The chain rule is used in the backpropagation algorithm for neural networks. Suppose an error function E depends on the activations $a_i^{(n)}$ of the units in the n^{th} (top) layer, where i labels the units, and these activities depend in turn on the activities $a_i^{(n-1)}$ in the previous layer and the weights $w_{ij}^{(n)}$ (representing the weight from the j^{th} unit of layer $n-1$ to the i^{th} unit of layer n). Then we can determine how the error will change by modifying a weight $w_{ij}^{(n)}$ by writing

$$\frac{dE}{dw_{ij}^{(n)}} = \frac{dE}{da_i^{(n)}}\frac{da_i^{(n)}}{dw_{ij}^{(n)}} \tag{0.35}$$

One can keep concatenating derivatives to determine how weights in lower layers should change, for example to determine how a change in a weight $w_{ij}^{(n-1)}$ from layer $n-2$ to layer $n-1$ will change the error, we write

$$\frac{dE}{dw_{ij}^{(n-1)}} = \sum_p \frac{dE}{da_p^{(n)}}\frac{da_p^{(n)}}{da_i^{(n-1)}}\frac{da_i^{(n-1)}}{dw_{ij}^{(n-1)}} \tag{0.36}$$

0.4 Deriving the Example Derivatives, and the Number e

Although we wrote down $\frac{de^t}{dt} = e^t$ above, it is useful to see precisely why this is true. Note first that the function z^t for some number z has the derivative $\frac{dz^t}{dt} = \lim_{\epsilon\to 0}\frac{z^{t+\epsilon}-z^t}{\epsilon} = \lim_{\epsilon\to 0} z^t\left(\frac{z^\epsilon - 1}{\epsilon}\right) = z^t \lim_{\epsilon\to 0}\frac{z^\epsilon-1}{\epsilon}$. So for any z, $\frac{dz^t}{dt} = k(z)z^t \propto z^t$, with the proportionality constant given by $k(z) = \lim_{\epsilon\to 0}\frac{z^\epsilon-1}{\epsilon}$. It's easy to see that $k(1) = 0$ and that $k(z)$ is an increasing function of z. For which value of z is $k(z) = 1$, so that $f(t) = z^t$ satisfies $\frac{df(t)}{dt} = f(t)$?

To answer, recall the number e, Euler's number, which has the value $2.71828\ldots$. It is the solution to the compound interest problem: suppose you pay back a loan with 100% interest after

1 year, so that at the end of the year you pay $(1+1)$ times the loan amount. You could instead accrue 50% interest every six months, so you would owe $(1+\frac{1}{2})^2$ times the loan amount. You could break it up into finer and finer bins, adding $1/N$ in interest N times. Taking the limit as N goes to infinity, this limiting amount of principle plus interest is e:

$$e = \lim_{N\to\infty}\left(1+\frac{1}{N}\right)^N \tag{0.37}$$

Thus $e^t = \lim_{N\to\infty}(1+\frac{1}{N})^{Nt}$ and by a change of variables, $p = Nt$, we can write this as $e^t = \lim_{p\to\infty}(1+\frac{t}{p})^p$. So $e^\epsilon = \lim_{p\to\infty}(1+\frac{\epsilon}{p})^p$. If this limit exists, then the infinite sequence of integers going to infinity has to give the same answer, so we can consider $\lim_{p\to\infty, p \text{ integer}}(1+\frac{\epsilon}{p})^p$. For integer p, it is easy to expand the polynomial: $(1+\frac{\epsilon}{p})^p = 1+p\frac{\epsilon}{p}+O(\epsilon^2) = 1+\epsilon+O(\epsilon^2)$ (here $O(\epsilon^2)$ means terms involving ϵ raised to a power 2 or higher). This shows that for the number e, $k(e)=1$, and so e^t is the function $f(t)$ that is its own derivative, $\frac{df(t)}{dt} = f(t)$.

Note that, knowing the derivative of e^t and thus, by the chain rule, that of $e^{\epsilon t}$, we can write the Taylor series $e^{\epsilon t} = 1+\epsilon t+O(\epsilon^2)$. This allows us to evaluate $k(z) = \lim_{\epsilon\to 0}\frac{z^\epsilon-1}{\epsilon} = \lim_{\epsilon\to 0}\frac{e^{\epsilon\log(z)}-1}{\epsilon} = \lim_{\epsilon\to 0}\frac{1+\epsilon\log(z)+O(\epsilon^2)-1}{\epsilon} = \log(z)$, that is, $\frac{dz^t}{dt} = \log(z)z^t$.

The other example derivatives of Section 0.1 can be derived as follows. The derivative of the log can be found by writing $e^{\log(x)} = x$; differentiating both sides with respect to x and using the chain rule on the left gives $\frac{d\log(x)}{dx}e^{\log(x)} = 1$, or $\frac{d\log(x)}{dx} = \frac{1}{x}$. Then to find the derivative of x^n, we write $x^n = e^{n\log(x)}$. We differentiate both sides and apply the chain rule to find $\frac{dx^n}{dx} = \frac{dn\log(x)}{dx}e^{n\log(x)} = \frac{n}{x}x^n = nx^{n-1}$.

To find the derivative of $\cos(x)$, we write $\frac{d\cos(x)}{dx} = \lim_{\epsilon\to 0}\frac{\cos(x+\epsilon)-\cos(x)}{\epsilon}$. We can use a trigonometric identity to write $\cos(x+\epsilon) = \cos(x)\cos(\epsilon)-\sin(x)\sin(\epsilon)$, so $\frac{d\cos(x)}{dx} = \lim_{\epsilon\to 0} -\cos(x)\frac{1-\cos(\epsilon)}{\epsilon} - \sin(x)\frac{\sin(\epsilon)}{\epsilon}$. Here we have to rely on knowing $\lim_{\epsilon\to 0}\frac{\sin(\epsilon)}{\epsilon} = 1$ and $\lim_{\epsilon\to 0}\frac{1-\cos(\epsilon)}{\epsilon} = 0$, which can be established by purely geometric analysis (but can also be seen to be true, circularly, if we already know the derivatives of cos and sin so that we can use their Taylor series expansion). This gives $\frac{d\cos(x)}{dx} = -\sin(x)$. Similarly, $\frac{d\sin(x)}{dx} = \lim_{\epsilon\to 0}\frac{\sin(x+\epsilon)-\sin(x)}{\epsilon}$, and $\sin(x+\epsilon) = \cos(x)\sin(\epsilon)+\sin(x)\cos(\epsilon)$, so $\frac{d\sin(x)}{dx} = \lim_{\epsilon\to 0} -\sin(x)\frac{1-\cos(\epsilon)}{\epsilon} + \cos(x)\frac{\sin(\epsilon)}{\epsilon} = \cos(x)$.

0.5 First-Order Linear Differential Equations of One Variable

A first-order differential equation is one that involves only first derivatives, *i.e.* no higher derivatives. A linear differential equation is one in which the dependencies on the unknown function and its derivatives are linear. Thus a first-order linear differential equation has the form $f'(t) = a(t)f(t) + b(t)$ (we switch to using the variable t instead of x, because it is more intuitive to think of these equations as describing evolution in time rather than in space). If $a(t) = 0$, the equation can be solved by integrating both sides to obtain $f(t) = \int dt\, b(t) + C$. We will be interested in this equation with $a(t) \neq 0$. (Some terminology: if there is no explicit time dependence, meaning that a and b are both time-independent constants, the equation is called *autonomous*; otherwise, it is *non-autonomous*. If $b(t) = 0$, so that all the terms in the equation are linear in f or f', it is called a *homogeneous* linear equation; otherwise, it is *non-homogeneous*. For nonlinear differential equations, the definition of homogeneous is a bit more complicated, see `https://en.wikipedia.org/wiki/Homogeneous_differential_equation`.)

The simplest such equation is autonomous and homogeneous:

$$\frac{df}{dt} = \pm f(t) \tag{0.38}$$

This has the solution

$$f(t) = f(0)e^{\pm t} \tag{0.39}$$

where the sign of the $\pm$ in the solution is the same as that in the differential equation, and $f(0)$ is the value of f at the initial time $t = 0$ (here and below, any arbitrary initial time t_0 can be substituted for 0, with the "$t - 0$" dependence (here, in $e^t = e^{t-0}$) replaced by $t - t_0$).

Using the chain rule to obtain $\frac{de^{at}}{dt} = ae^{at}$, we see that the equation

$$\frac{df}{dt} = af(t) \tag{0.40}$$

has the solution[3]

$$f(t) = f(0)e^{at} \tag{0.41}$$

If we let a depend on t, the equation

$$\frac{df}{dt} = a(t)f(t) \tag{0.42}$$

has solution[4]

$$f(t) = f(0)e^{\int_0^t dt' a(t')} \tag{0.47}$$

That this is the solution to Eq. 0.42 can be seen by the chain rule:

$$\frac{d\left(f(0)e^{\int_0^t dt' a(t')}\right)}{dt} = f(0)\frac{d\left(e^{\int_0^t dt' a(t')}\right)}{d\left(\int_0^t dt' a(t')\right)}\frac{d\left(\int_0^t dt' a(t')\right)}{dt} = f(t)\frac{d\left(\int_0^t dt' a(t')\right)}{dt} = a(t)f(t) \tag{0.48}$$

To evaluate the last derivative, note that adding ϵ to t just adds $a(t)\epsilon$ to the value of the integral, to lowest order in ϵ, so the derivative (value of function at $t + \epsilon$ minus value at t, divided by ϵ) is just $a(t)$. That is, differentiating an integral with respect to its upper limit (here, t) just produces the integrand evaluated at the upper limit (here, $a(t)$).

Finally, we consider the inhomogeneous case, meaning that $b(t) \neq 0$. First, we consider the autonomous case, in which both a and b are constants:

$$\frac{df}{dt} = af(t) + b \tag{0.49}$$

[3]If time 0 is replaced by an initial time t_0, the solution in Eq. 0.41 can be written $f(t_0)e^{a(t-t_0)}$ (note that e^{at} was implicitly $e^{a(t-0)}$) and that in Eq. 0.47 can be written $f(t_0)e^{\int_{t_0}^t dt' a(t')}$.

[4]This can be derived by proceeding from Eq. 0.42 to write

$$\frac{df}{f(t)} = a(t)dt \tag{0.43}$$

$$\int_{f(0)}^{f(t)} \frac{df}{f} = \int_0^t dt'\, a(t') \tag{0.44}$$

$$\log(f(t)) - \log(f(0)) = \int_0^t dt'\, a(t') \tag{0.45}$$

$$f(t)/f(0) = e^{\int_0^t dt'\, a(t')} \tag{0.46}$$

This is an example of the method of *separation of variables*, which can be used to solve any linear homogeneous equation: all the terms involving f are on one side of the equation with df, and all the terms depending on t (except $f(t)$) are on the right side with dt, allowing each side to be integrated.

We can see that this equation has a fixed point (meaning a point where $\frac{df}{dt} = 0$) where $f(t) = -b/a \equiv f_{FP}$. Then the solution is[5]

$$f(t) = f(0)e^{at} + f_{FP}(1 - e^{at}) = f_{FP} + e^{at}\left(f(0) - f_{FP}\right) \tag{0.50}$$

as can be checked by taking the derivative $\frac{df}{dt}$. For $a < 0$, this can be understood as the initial condition exponentially decaying away while the system exponentially decays from 0 to the final condition (the fixed point), or as the difference between the initial and final conditions exponentially decaying away. For $a > 0$, this represents exponential growth away from the fixed point.

Next, we consider dependence of b on t,

$$\frac{df}{dt} = af(t) + b(t) \tag{0.51}$$

In this case the solution can be written[6]

$$f(t) = f(0)e^{at} + \int_0^t dt'\, e^{a(t-t')}b(t') \tag{0.52}$$

This can again be seen by differentiating. We have to note that the derivative of the integral with respect to t gives two terms: (1) differentiating with respect to the upper limit t gives the integrand evaluated at $t' = t$, which is $b(t)$; (2) differentiating with respect to the t in the exponential brings down a factor of a, that is, it gives the integral multiplied by a. It is also worth checking that when $b(t)$ is constant, $b(t) = b$, this gives the solution in Eq. 0.50. Note that the solution in Eq. 0.52 can be intuitively understood as $b(t')dt'$ providing a new source or initial condition at time t', which decays away exponentially with increasing time (it is multiplied by $e^{a(t-t')}$); and all of these decayed sources are added up (integrated), along with adding the decayed initial condition $f(0)$, to give the current state.

Finally, we consider the case in which all elements are time-dependent:

$$\frac{df}{dt} = a(t)f(t) + b(t) \tag{0.53}$$

Then the solution is[7]

$$f(t) = f(0)e^{\int_0^t dt' a(t')} + \int_0^t dt'\, e^{\int_{t'}^t dq\, a(q)}b(t') \tag{0.54}$$

Again, this can be verified by differentiating with respect to t: the derivatives of the integrals in the exponentials just multiply each term by $a(t)$; while the derivative of the outer integral gives the integrand at $t' = t$, which is $b(t)$. As before, it is useful to verify that making $a(q)$ constant, $a(q) = a$, reduces this solution to that of Eq. 0.52.

[5] For a derivation for time-dependent b, see footnote 6; the same derivation gives Eq. 0.50 when b is constant.

[6] The solution can be derived as follows. We know the homogeneous solution is proportional to e^{at}. The trick now is to guess a solution of the form $f(t) = e^{at}g(t)$. Taking the derivative gives $\frac{d}{dt}f(t) = af(t) + e^{at}\frac{d}{dt}g(t)$. Matching this to $\frac{d}{dt}f(t) = af(t) + b(t)$ yields $\frac{d}{dt}g(t) = e^{-at}b(t)$; integration then gives $g(t) = \int_0^t dt'\, e^{-at'}b(t') + g(0)$. Setting $t = 0$ in $f(t) = e^{at}g(t)$ gives $g(0) = f(0)$. Putting this all together gives $f(t) = \int_0^t dt'\, e^{a(t-t)'}b(t') + e^{at}f(0)$, which is Eq. 0.52.

[7] The solution can be derived much as in footnote 6. We know the homogeneous solution is proportional to $e^{\int_0^t dt'\, a(t')}$. We guess a solution of the form $f(t) = e^{\int_0^t dt'\, a(t')}g(t)$; note that taking $t = 0$ then gives $f(0) = g(0)$. Taking the derivative gives $\frac{d}{dt}f(t) = a(t)f(t) + e^{\int_0^t dt'\, a(t')}\frac{d}{dt}g(t)$. Matching this to $\frac{d}{dt}f(t) = a(t)f(t) + b(t)$ yields $\frac{d}{dt}g(t) = e^{-\int_0^t dt'\, a(t')}b(t)$; integration then gives $g(t) = \int_0^t dt'\, e^{-\int_0^{t'} dq\, a(q)}b(t') + g(0)$. Putting this all together, using $g(0) = f(0)$, gives $f(t) = e^{\int_0^t dt'\, a(t')}\left(\int_0^t dt'\, e^{-\int_0^{t'} dq\, a(q)}b(t') + f(0)\right) = f(0)e^{\int_0^t dt'\, a(t')} + \int_0^t dt'\, e^{\int_{t'}^t dq\, a(q)}b(t')$, which is Eq. 0.54.